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ORIGINAL PAPER

Real wavefunction from Generalised Hamiltonian Schrodinger Equation in quantum phase space via HOA (Heaviside Operational Ansatz): exact analytical results

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Abstract This note reports the decomposition of a complex wavefunction into real and imaginary parts via the Heaviside Operational Ansatz (HOA) in the Quantum Phase Space Representation; this for a Complex Schrodinger Equation (CSE) generated by a given Generalised Hamiltonian. Consequently, the CSE is re-cast as a two-component Real Schrodinger Equation (RSE) system that couples the real and imaginary parts of the complex wavefunction from the original CSE. By way of the HOA, the resulting real and imaginary parts of the complex wavefunction may be expressed in terms of each other: the 'upshot' being the de-coupling of the two-component system into a separate new RSE for the real part and a new separate RSE for the imaginary part, of the original CSE complex wavefunction. It follows that the new de-coupled RSE for the real (respectively imaginary) part of the original complex wavefunction is 'informationally-equivalent' to the new de-coupled RSE for the imaginary (respectively real) part of the original complex wavefunction. Moreover, each of these RSEs for the aforementioned real wavefunctions is also 'informationally-equivalent' to the original CSE and it's complex wavefunction. What's more, consequent of HOA, the real wavefunctions each have exact analytical expression in quadrature (integral form).

Keywords Real wavefunction quantum dynamics · Heaviside Operational Ansatz · Exact analytical solution · Quantum phase space · Generalised complex multitime Hamiltonian · Real Schrodinger Equation

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1 Motivation

Since Erwin Schrodinger first enunciated his revolutionary equation in 1926, a fundamental query has been, ‘Can the complex Schrodinger equation (and it’s complex wavefunction solution) be expressed as an ‘informationally-equivalent’ real ‘Schrodinger-like’ equation with corresponding real wavefunction?’ (In particular, see the pioneering research of R.L.W. Chen, Journal of Mathematical Physics **30**, 83 (1989) and the references therein). If such an alternative expression of Quantum Dynamics were possible, it would constitute a disruptive mathematical and physical paradigm shift in the way Quantum Theory may be studied.

Along this march of reason, herein the decomposition of a complex wavefunction into real and imaginary parts in the QPSR (Quantum Phase Space Representation) (ref. [1]) is advanced, via the HOA (Heaviside Operational Ansatz) (e.g., refs. [2–8]). This for a CSE (Complex Schrodinger Equation) generated by a given GH (Generalised Hamiltonian); the GH is a given complex function of its arguements. Consequently, the CSE is re-cast as a two-component RSE (Real Schrodinger Equation) system that couples the real and imaginary parts of the complex wavefunction from the original CSE. Via HOA, the resulting real and imaginary parts of the complex wavefunction are resolved in terms of each other. Thus: the de-coupling of the two-component system into a separate new RSE for the real part and a new separate RSE for the imaginary part, of the original CSE complex wavefunction. It follows that the new de-coupled RSE for the real (respectively imaginary) part of the original complex wavefunction is ‘informationally-equivalent’ to the new de-coupled RSE for the imaginary (respectively real) part of the original complex wavefunction. Moreover, each of these new RSEs for the aforementioned real wavefunctions is also ‘informationally-equivalent’ to the original CSE and it’s complex wavefunction. What’s more, by way of HOA, the consequent real wavefunctions are then each expressed exactly in analytical quadrature (integral form).

The GH which generates the CSE having complex wavefunction, as well as the alternative formulation via an RSE system coupling the real and imaginary parts of said complex wavefunction, may thus be expressed either in purely complex form

$$\begin{aligned}&\left(\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)+i\hat{H}_{im}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)\right)\left(\Psi_{re}+i\Psi_{im}\right)\\&\quad=i\hbar\partial_t\left(\Psi_{re}+i\Psi_{im}\right)\end{aligned}\tag{1}$$

or in purely real form

$$\begin{aligned}&\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)\Psi_{re}-\hat{H}_{im}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)\\&\quad\Psi_{im}=-\hbar\partial_t\Psi_{im}\\&\hat{H}_{im}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)\Psi_{re}+\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)\\&\quad\Psi_{im}=\hbar\partial_t\Psi_{re}\end{aligned}\tag{2}$$

For convenience, we begin with a recap of the basic HOA construction following from Simpao [2].

2 Recap of HOA

Here are the basic relations; x, p, t are respectively the configuration space position, momentum and time variables. The hat (^) denotes the operators, with H and Ψ denoting the Hamiltonian and wavefunction of the QPSR, respectively. Also, the α, γ, are otherwise free parameters as specified in ref. [1].

$$
\begin{aligned}
&H(x,p,t) \to \hat{H}\left(\hat{x},\hat{p},t\right) = \hat{H}\left(i\hbar\partial_p + \alpha x, -i\hbar\partial_x + \gamma p, t\right), \ni \alpha+\gamma = 1\\
&x \to \hat{x} \equiv i\hbar\partial_p + \alpha x,\ p \to \hat{p} \equiv -i\hbar\partial_x + \gamma p,\ t \to t = t\\
&(x_1,\ldots,x_n) \to \left(\hat{x}_1,\ldots,\hat{x}_n\right) = \left(i\hbar\partial_{p_1} + \alpha_1 x_1,\ldots,i\hbar\partial_{p_n} + \alpha_n x_n\right),\\
&\quad \ni \alpha_j + \gamma_j = 1,\ j = 1,\ldots,n\\
&(p_1,\ldots,p_n) \to \left(\hat{p}_1,\ldots,\hat{p}_n\right) = \left(-i\hbar\partial_{x_1} + \gamma_1 p_1,\ldots,-i\hbar\partial_{x_n} + \gamma_n p_n\right)\\
&H(x_1,\ldots,x_n;p_1,\ldots,p_n;t) \to \hat{H}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)\\
&\quad \equiv \hat{H}\left(i\hbar\partial_{p_1} + \alpha_1 x_1,\ldots,i\hbar\partial_{p_n} + \alpha_n x_n; -i\hbar\partial_{x_1} + \gamma_1 p_1,\ldots,-i\hbar\partial_{x_n} + \gamma_n p_n;t\right)
\end{aligned}
\tag{3}
$$

Now with the following properties of Heaviside operational methods via Laplace transforms

$$
\begin{aligned}
L_{y\to z}\left[f(y)\right] &= \int_{y_0}^{\infty} f(y)\, e^{-yz} dy = \breve{f}(z)\\
L^{-1}_{z\to y}\left[\breve{f}(z)\right] &= \frac{1}{2\pi i}\oint_{\partial} \breve{f}(z)\, e^{yz} dz = f(y)\\
L^{-1}_{z\to y}\left[\breve{f}(z)\right] &= \frac{1}{2\pi i}\oint_{\partial} \breve{f}(z)\, e^{yz} dz = f(y) = \breve{f}\left(D_y\right) U(y)\\
L^{-1}_{z\to y}\left[\breve{f}_1(z)\,\breve{f}_2(z)\right] &= f_1(y) * f_2(y) = \int_{y_0}^{y} f_1(y-u)\, f_2(u)\, du\\
&= \breve{f}_1\left(D_y\right)\breve{f}_2\left(D_y\right) U(y) = \breve{f}_1\left(D_y\right) f_2(y)
\end{aligned}
\tag{4}
$$

where $U(y)$ is the Heaviside Unit Step function

$$
\begin{aligned}
&L_{(y_1,\ldots,y_n)\to(z_1,\ldots,z_n)}\left[f(y_1,\ldots,y_n)\right]\\
&= \underbrace{\int_{y_{0n}}^{\infty}\ldots\int_{y_{01}}^{\infty}}_{n} f(y_1,\ldots,y_n)\, e^{-\sum_{j=1}^{n} y_j z_j} dy_1\ldots dy_n = \breve{f}(z_1,\ldots,z_n)\\
&L^{-1}_{(z_1,\ldots,z_n)\to(y_1,\ldots,y_n)}\left[\breve{f}(z_1,\ldots,z_n)\right]
\end{aligned}
$$

$$= \left(\frac{1}{2\pi i}\right)^n \oint_{\partial^n} \breve{f}(z_1,\ldots,z_n)\, e^{\sum_{j=1}^{n} y_j z_j} dz_1 \ldots dz_n = f(y_1,\ldots,y_n)$$

$$L^{-1}_{(z_1,\ldots,z_n)\to(y_1,\ldots,y_n)}\left[\breve{f}_1(z_1,\ldots,z_n)\,\breve{f}_2(z_1,\ldots,z_n)\right]$$
$$= f_1(y_1,\ldots,y_n) \underbrace{*}_{(y_1,\ldots,y_n)} f_2(y_1,\ldots,y_n)$$
$$= \int_{y_{0n}}^{y_n} \underbrace{\ldots}_{n} \int_{y_{01}}^{y_1} f_1\left(y_1-y_1',\ldots,y_n-y_n'\right) f_2\left(y_1',\ldots,y_n'\right) dy_1' \ldots dy_n'$$
$$= \breve{f}_1\left(\partial_{y_1},\ldots,\partial_{y_n}\right) \breve{f}_2(y_1,\ldots,y_n)\, U(y_1,\ldots,y_n)$$
$$= \breve{f}_1\left(\partial_{y_1},\ldots,\partial_{y_n}\right) f_2(y_1,\ldots,y_n)$$

where the zero-subscripted variables (e.g., y_0) are the arbitrarily specified lower limits of integration.

With the phase-space convolution

$$f_1(x_1,\ldots,x_n;p_1,\ldots,p_n) \underbrace{*}_{(x_1,\ldots,x_n;p_1,\ldots,p_n)} f_2(x_1,\ldots,x_n;p_1,\ldots,p_n)$$
$$= \int_{x_{0n}}^{x_n} \underbrace{\ldots}_{n} \int_{x_{01}}^{x_1} \int_{p_{0n}}^{p_n} \underbrace{\ldots}_{n} \int_{p_{01}}^{p_1} f_1\left(x_1-x_1',\ldots,x_n-x_n';p_1-p_1',\ldots,p_n-p_n'\right)$$
$$\times f_2\left(x_1',..,x_n';p_1',..,p_n'\right) dx_1'\ldots dx_n' dp_1'\ldots dp_n' \tag{5}$$

Lower bounds of respective phase space co-ordinates: $(x_{10},\ldots,x_{n0};p_{10},\ldots,p_{n0})$ Also the transform relation

$$L_{z\to y}\left[\breve{f}(az-b)\right] = \frac{1}{a} e^{\frac{by}{a}} f\left(\frac{y}{a}\right)$$
$$L^{-1}_{(z_1,\ldots,z_n)\to(y_1,\ldots,y_n)}\left[\breve{f}(a_1 z_1-b_1,\ldots,a_n z_n-b_n)\right]$$
$$= \prod_{j=1}^{n} \frac{1}{a_j} e^{\frac{b_j}{a_j} y_j} f\left(\frac{y_1}{a_1},\ldots,\frac{y_n}{a_n}\right) \tag{6}$$

From (3) the wave equation becomes

$$\hat{H}\begin{pmatrix} i\hbar\partial_{p_1}+\alpha_1 x_1,\ldots,i\hbar\partial_{p_n}+\alpha_n x_n; \\ -i\hbar\partial_{x_1}+\gamma_1 p_1,\ldots,-i\hbar\partial_{x_n}+\gamma_n p_n;t \end{pmatrix} \Psi(x_1,\ldots,x_n;p_1,\ldots,p_1;t)$$
$$= i\hbar\partial_t \Psi(x_1,\ldots,x_n;p_1,\ldots,p_1;t) \tag{7}$$

Applying the convolution identity and multivariable inverse transform of (4), the phase space convolution of (5) and relation (6) yields

$$\left[L^{-1}_{\left(\begin{smallmatrix}(\partial_{x_1},\ldots,\partial_{x_n})\\ \to(x_1,\ldots,x_n)\end{smallmatrix}\right)} \left[L^{-1}_{\left(\begin{smallmatrix}(\partial_{p_1},\ldots,\partial_{p_n})\\ \to(p_1,\ldots,p_n)\end{smallmatrix}\right)} \left[\hat{H}\begin{pmatrix} i\hbar\partial_{p_1}+\alpha_1 x_1,\ldots,i\hbar\partial_{p_n}+\alpha_n x_n; \\ -i\hbar\partial_{x_1}+\gamma_1 p_1,\ldots,-i\hbar\partial_{x_n}+\gamma_n p_n;t \end{pmatrix} \right]\right] \underset{(x_1,\ldots,x_n;p_1,\ldots,p_n)}{*} \Psi(x_1,\ldots,x_n;p_1,\ldots,p_n;t) \right]$$
$$\equiv \hat{H}\begin{pmatrix} i\hbar\partial_{p_1}+\alpha_1 x_1,\ldots,i\hbar\partial_{p_n}+\alpha_n x_n; \\ -i\hbar\partial_{x_1}+\gamma_1 p_1,\ldots,-i\hbar\partial_{x_n}+\gamma_n p_n;t \end{pmatrix}\Psi(x_1,\ldots,x_n;p_1,\ldots,p_n;t) \quad (8)$$

Applying (8) to (7) with the convolution identities in (4) and transforming

$$L_{\left(\begin{smallmatrix}(p_1,\ldots,p_n)\\ \to(\bar{p}_1,\ldots,\bar{p}_n)\end{smallmatrix}\right)}\left[L_{\left(\begin{smallmatrix}(x_1,\ldots,x_n)\\ \to(\bar{x}_1,\ldots,\bar{x}_n)\end{smallmatrix}\right)}\left[\begin{matrix}\hat{H}\begin{pmatrix} i\hbar\partial_{p_1}+\alpha_1 x_1,\ldots,i\hbar\partial_{p_n}+\alpha_n x_n; \\ -i\hbar\partial_{x_1}+\gamma_1 p_1,\ldots,-i\hbar\partial_{x_n}+\gamma_n p_n;t \end{pmatrix}\Psi(x_1,\ldots,x_n;p_1,\ldots,p_n;t) \\ = i\hbar\partial_t\Psi(x_1,\ldots,x_n;p_1,\ldots,p_1;t)\end{matrix}\right]\right]$$
$$\equiv$$
$$L_{\left(\begin{smallmatrix}(p_1,\ldots,p_n)\\ \to(\bar{p}_1,\ldots,\bar{p}_n)\end{smallmatrix}\right)}\left[L_{\left(\begin{smallmatrix}(x_1,\ldots,x_n)\\ \to(\bar{x}_1,\ldots,\bar{x}_n)\end{smallmatrix}\right)}\left[\begin{matrix} L^{-1}_{\left(\begin{smallmatrix}(\partial_{x_1},\ldots,\partial_{x_n})\\ \to(x_1,\ldots,x_n)\end{smallmatrix}\right)} \left[L^{-1}_{\left(\begin{smallmatrix}(\partial_{p_1},\ldots,\partial_{p_n})\\ \to(p_1,\ldots,p_n)\end{smallmatrix}\right)} \left[\hat{H}\begin{pmatrix} i\hbar\partial_{p_1}+\alpha_1 x_1,\ldots,i\hbar\partial_{p_n}+\alpha_n x_n; \\ -i\hbar\partial_{x_1}+\gamma_1 p_1,\ldots,-i\hbar\partial_{x_n}+\gamma_n p_n;t \end{pmatrix} \right]\right] \\ \underset{(x_1,\ldots,x_n;p_1,\ldots,p_n)}{*} \Psi(x_1,\ldots,x_n;p_1,\ldots,p_n;t) \\ = i\hbar\partial_t\Psi(x_1,\ldots,x_n;p_1,\ldots,p_n;t)\end{matrix}\right]\right]$$
$$\equiv$$
$$\hat{H}\begin{pmatrix} i\hbar\partial_{p_1}+\alpha_1 x_1,\ldots,i\hbar\partial_{p_n}+\alpha_n x_n; \\ -i\hbar\partial_{x_1}+\gamma_1 p_1,\ldots,-i\hbar\partial_{x_n}+\gamma_n p_n;t \end{pmatrix}_{\left(\begin{smallmatrix}\partial_{p_1}\mapsto\bar{p}_1,\\ \ldots,\partial_{p_n}\mapsto\bar{p}_n;\\ \partial_{x_1}\mapsto\bar{x}_1,\\ \ldots,\partial_{x_n}\mapsto\bar{x}_n\end{smallmatrix}\right)} \breve{\Psi}(\bar{x}_1,\ldots,\bar{x}_n;\bar{p}_1,\ldots,\bar{p}_n;t) = i\hbar\partial_t\breve{\Psi}(\bar{x}_1,\ldots,\bar{x}_n;\bar{p}_1,\ldots,\bar{p}_n;t)$$
$$\equiv$$
$$\hat{H}\begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1,\ldots,i\hbar\bar{p}_n+\alpha_n x_n; \\ -i\hbar\bar{x}_1+\gamma_1 p_1,\ldots,-i\hbar\bar{x}_n+\gamma_n p_n;t \end{pmatrix}\breve{\Psi}(\bar{x}_1,\ldots,\bar{x}_n;\bar{p}_1,\ldots,\bar{p}_n;t) = i\hbar\partial_t\breve{\Psi}(\bar{x}_1,\ldots,\bar{x}_n;\bar{p}_1,\ldots,\bar{p}_n;t) \quad (9)$$

Hence the wavefunction in phase space may be analytically expressed in exact quadratures, by inverse transforming the above solution $\breve{\psi}\,(\bar{x}_1,\ldots,\bar{x}_n;\ \bar{p}_1,\ldots,\bar{p}_n;\ t)$ of (9) as

$$\hat{H}\begin{pmatrix} i\hbar\overline{p}_1+\alpha_1 x_1,\ldots.,i\hbar\overline{p}_n+\alpha_n x_n; \\ -i\hbar\overline{x}_1+\gamma_1 p_1,\ldots.,-i\hbar\overline{x}_n+\gamma_n p_n;t \end{pmatrix}\breve{\Psi}(\overline{x}_1,\ldots.,\overline{x}_n;\overline{p}_1,\ldots.,\overline{p}_n;t)=i\hbar\partial_t\breve{\Psi}(\overline{x}_1,\ldots.,\overline{x}_n;\overline{p}_1,\ldots.,\overline{p}_n;t)$$

$$\Psi(x_1,\ldots.,x_n;p_1,\ldots.,p_n;t)=L^{-1}_{\begin{pmatrix}(\overline{x}_1,\ldots.,\overline{x}_n)\\ \to(x_1,\ldots.,x_n)\end{pmatrix}}\left[L^{-1}_{\begin{pmatrix}(\overline{p}_1,\ldots.,\overline{p}_n)\\ \to(p_1,\ldots.,p_n)\end{pmatrix}}\left[\begin{matrix} e^{\frac{-i}{\hbar}\int_0^t \hat{H}\begin{pmatrix} i\hbar\overline{p}_1+\alpha_1x_1,\ldots.,i\hbar\overline{p}_n+\alpha_nx_n; \\ -i\hbar\overline{x}_1+\gamma_1p_1,\ldots.,-i\hbar\overline{x}_n+\gamma_np_n;u\end{pmatrix}du} \\ \times \\ \breve{\Psi}_0(\overline{x}_1,\ldots.,\overline{x}_n;\overline{p}_1,\ldots.,\overline{p}_n;t=0)\end{matrix}\right]\right] \quad (10)$$

A "New Twist" on this was recently (13 May 2010), discovered by Simpao (refs. [5–8]): namely an alternative exact analytical quadrature expression for the quantum phase space wave function. By observing from (10) that

$$\Psi(x_1,\ldots.,x_n;p_1,\ldots.,p_n;t)=L^{-1}_{\begin{pmatrix}(\overline{x}_1,\ldots.,\overline{x}_n)\\ \to(x_1,\ldots.,x_n); \\ (\overline{p}_1,\ldots.,\overline{p}_n)\\ \to(p_1,\ldots.,p_n)\end{pmatrix}}\begin{pmatrix}\breve{\Psi}(\overline{x}_1,\ldots.,\overline{x}_n;\overline{p}_1,\ldots.,\overline{p}_n;t) \\ =\breve{\Psi}_0(\overline{x}_1,\ldots.,\overline{x}_n;\overline{p}_1,\ldots.,\overline{p}_n;t=0)e^{\frac{-i}{\hbar}\int_0^t \hat{H}\begin{pmatrix} i\hbar\overline{p}_1+\frac{1}{2}x_1,\ldots.,i\hbar\overline{p}_n+\frac{1}{2}x_n; \\ -i\hbar\overline{x}_1+\frac{1}{2}p_1,\ldots.,-i\hbar\overline{x}_n+\frac{1}{2}p_n;u\end{pmatrix}du}\end{pmatrix} \quad (10a)$$

Theory of Shifted Taylor Series yields

$$e^{\frac{-i}{\hbar}\int_0^t \hat{H}\begin{pmatrix} i\hbar\bar{p}_1+\frac{1}{2}x_1,\ldots.,i\hbar\bar{p}_n+\frac{1}{2}x_n; \\ -i\hbar\bar{x}_1+\frac{1}{2}p_1,\ldots.,-i\hbar\bar{x}_n+\frac{1}{2}p_n;u\end{pmatrix}du}$$
$$= e^{i\hbar\bar{p}_1\partial_{\frac{1}{2}x_1}\cdots i\hbar\bar{p}_n\partial_{\frac{1}{2}x_n}}e^{-i\hbar\bar{x}_1\partial_{\frac{1}{2}p_1}\cdots -i\hbar\bar{x}_n\partial_{\frac{1}{2}p_n}}\left(e^{\frac{-i}{\hbar}\int_0^t \hat{H}\begin{pmatrix}\frac{1}{2}x_1,\ldots.,\frac{1}{2}x_n; \\ \frac{1}{2}p_1,\ldots.,\frac{1}{2}p_n;u\end{pmatrix}du}\right) \quad (10b)$$

and Heaviside's Alternative to the Convolution developed above yields

$$e^{i\hbar\overline{p}_1\partial_{\frac{1}{2}x_1}\cdots i\hbar\overline{p}_n\partial_{\frac{1}{2}x_n}}e^{-i\hbar\overline{x}_1\partial_{\frac{1}{2}p_1}\cdots -i\hbar\overline{x}_n\partial_{\frac{1}{2}p_n}}\left(e^{\frac{-i}{\hbar}\int_0^t \hat{H}\begin{pmatrix}\frac{1}{2}x_1,\ldots.,\frac{1}{2}x_n; \\ \frac{1}{2}p_1,\ldots.,\frac{1}{2}p_n;u\end{pmatrix}du}\right)$$
$$\equiv \delta\left(\tfrac{1}{2}x_1+i\hbar\overline{p}_1,\ldots.,\tfrac{1}{2}x_n+i\hbar\overline{p}_n\right)\delta\left(\tfrac{1}{2}p_1-i\hbar\overline{x}_1,\ldots.,\tfrac{1}{2}p_n-i\hbar\overline{x}_n\right)\underset{\begin{pmatrix}\frac{1}{2}x_1,\ldots.,\frac{1}{2}x_n; \\ \frac{1}{2}p_1,\ldots.,\frac{1}{2}p_n\end{pmatrix}}{*}e^{\frac{-i}{\hbar}\int_0^t \hat{H}\begin{pmatrix}\frac{1}{2}x_1,\ldots.,\frac{1}{2}x_n; \\ \frac{1}{2}p_1,\ldots.,\frac{1}{2}p_n;u\end{pmatrix}du} \quad (10c)$$

Converting from the Laplace transform image variables to their Fourier counterparts

$$\begin{aligned}
&(\bar{p}_1,\ldots,\bar{p}_n)=(ip_{\omega 1},\ldots,ip_{\omega n})\,,\ (\bar{x}_1,\ldots,\bar{x}_n)=(ix_{\omega 1},\ldots,ix_{\omega n})\\
&\breve{\Psi}(\bar{x}_1,\ldots,\bar{x}_n;\bar{p}_1,\ldots,\bar{p}_n;t)=\breve{\Psi}(ix_{\omega 1},\ldots,ix_{\omega n};ip_{\omega 1},\ldots,ip_{\omega n};t)\\
&\delta\left(\tfrac{1}{2}x_1+i\hbar\bar{p}_1,\ldots,\tfrac{1}{2}x_n+i\hbar\bar{p}_n\right)\delta\left(\tfrac{1}{2}p_1-i\hbar\bar{x}_1,\ldots,\tfrac{1}{2}p_n-i\hbar\bar{x}_n\right)\\
&\equiv\delta\left(\tfrac{1}{2}x_1-\hbar p_{\omega 1},\ldots,\tfrac{1}{2}x_n-\hbar p_{\omega n}\right)\delta\left(\tfrac{1}{2}p_1+\hbar x_{\omega 1},\ldots,\tfrac{1}{2}p_n+\hbar x_{\omega n}\right)\\
&\delta\left(\tfrac{1}{2}x_1-\hbar p_{\omega 1},\ldots,\tfrac{1}{2}x_n-\hbar p_{\omega n}\right)\delta\left(\tfrac{1}{2}p_1+\hbar x_{\omega 1},\ldots,\tfrac{1}{2}p_n+\hbar x_{\omega n}\right)\underset{\begin{pmatrix}\frac{1}{2}x_1,\ldots,\frac{1}{2}x_n;\\ \frac{1}{2}p_1,\ldots,\frac{1}{2}p_n\end{pmatrix}}{\underbrace{*}}\; e^{\frac{-i}{\hbar}\int_0^t \hat{H}\begin{pmatrix}\frac{1}{2}x_1,\ldots,\frac{1}{2}x_n;\\ \frac{1}{2}p_1,\ldots,\frac{1}{2}p_n;u\end{pmatrix}du}
\end{aligned} \tag{10d}$$

Now the Inverse Fourier Transform of this is

$$\begin{aligned}
&F^{-1}_{\substack{(x_{\omega 1},\ldots,x_{\omega n})\\ \mapsto(x_1,\ldots,x_n)\\ (p_{\omega 1},\ldots,p_{\omega n})\\ \mapsto(p_1,\ldots,p_n)}}\left(\begin{array}{l}\delta\left(\tfrac{1}{2}x_1-\hbar p_{\omega 1},\ldots,\tfrac{1}{2}x_n-\hbar p_{\omega n}\right)\delta\left(\tfrac{1}{2}p_1+\hbar x_{\omega 1},\ldots,\tfrac{1}{2}p_n+\hbar x_{\omega n}\right)\\ \underset{\begin{pmatrix}\frac{1}{2}x_1,\ldots,\frac{1}{2}x_n;\\ \frac{1}{2}p_1,\ldots,\frac{1}{2}p_n\end{pmatrix}}{\underbrace{*}}\; e^{\frac{-i}{\hbar}\int_0^t \hat{H}\begin{pmatrix}\frac{1}{2}x_1,\ldots,\frac{1}{2}x_n;\\ \frac{1}{2}p_1,\ldots,\frac{1}{2}p_n;u\end{pmatrix}du}\end{array}\right)\\
&=\left(\frac{1}{2\pi\hbar^2}\right)^n \underset{\begin{pmatrix}\frac{1}{2}x_1,\ldots,\frac{1}{2}x_n;\\ \frac{1}{2}p_1,\ldots,\frac{1}{2}p_n\end{pmatrix}}{\underbrace{*}}\; e^{\frac{-i}{\hbar}\int_0^t \hat{H}\begin{pmatrix}\frac{1}{2}x_1,\ldots,\frac{1}{2}x_n;\\ \frac{1}{2}p_1,\ldots,\frac{1}{2}p_n;u\end{pmatrix}du}\\
&=\left(\frac{1}{2\pi\hbar^2}\right)^n \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n} e^{\frac{-i}{\hbar}\int_0^t \hat{H}\begin{pmatrix}\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\ldots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\\ \frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\ldots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u\end{pmatrix}du}\, d\nu_{\frac{1}{2}x_1}\,d\nu_{\frac{1}{2}p_1}\cdots d\nu_{\frac{1}{2}x_n}\,d\nu_{\frac{1}{2}p_n}
\end{aligned} \tag{10e}$$

Hence

$$\begin{aligned}
&L^{-1}_{\begin{pmatrix}(\bar{x}_1,\ldots,\bar{x}_n)\\ \to(x_1,\ldots,x_n);\\ (\bar{p}_1,\ldots,\bar{p}_n)\\ \to(p_1,\ldots,p_n)\end{pmatrix}}\left(e^{\frac{-i}{\hbar}\int_0^t \hat{H}\begin{pmatrix}i\hbar\bar{p}_1+\frac{1}{2}x_1,\ldots,i\hbar\bar{p}_n+\frac{1}{2}x_n;\\ -i\hbar\bar{x}_1+\frac{1}{2}p_1,\ldots,-i\hbar\bar{x}_n+\frac{1}{2}p_n;u\end{pmatrix}du}\right)\\
&=\left(\frac{1}{2\pi\hbar^2}\right)^n \int_{-\infty}^{\infty}\underbrace{\cdots}_{n}\int_{-\infty}^{\infty} e^{\frac{-i}{\hbar}\int_0^t \hat{H}\begin{pmatrix}\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\ldots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\\ \frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\ldots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u\end{pmatrix}du}\\
&\quad\times d\nu_{\frac{1}{2}x_1}d\nu_{\frac{1}{2}p_1}\cdots d\nu_{\frac{1}{2}x_n}d\nu_{\frac{1}{2}p_n}
\end{aligned} \tag{10f}$$

Thus the quantum phase space wavefunction with initial condition for the dynamics may be expressed in the direct real $(x_1,\ldots,x_n;p_1,\ldots,p_n)$ generalised coordinates and momenta.

$$\Psi\left(x_1,\ldots,x_n;p_1,\ldots,p_n;t\right) = \left(\begin{array}{l} \Psi_0\left(x_1,\ldots,x_n;p_1,\ldots,p_n;t=0\right) \\ \underset{\substack{\left(x_1,\ldots,x_n;\right)\\ \left(p_1,\ldots,p_n\right)}}{*}\left(\frac{1}{2\pi\hbar^2}\right)^n \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n} e^{\frac{-i}{\hbar}\int_0^t \hat{H}\left(\begin{array}{l}\alpha_1 x_1-\nu_{\alpha_1 x_1},\\ \ldots,\alpha_n x_n-\nu_{\alpha_n x_n};\\ \gamma_1 p_1-\nu_{\gamma_1 p_1},\\ \ldots,\gamma_n p_n-\nu_{\gamma_n p_n};u\end{array}\right)du} d\nu_{\alpha_1 x_1}\, d\nu_{\gamma_1 p_1}\cdots d\nu_{\alpha_n x_n}\, d\nu_{\gamma_n p_n} \end{array}\right) \tag{10g}$$

It is interesting to note that the canonical choice of $\alpha = \gamma = \frac{1}{2}$ thusly $\hat{H}\left(i\hbar\partial_p + \frac{x}{2}, -i\hbar\partial_x + \frac{p}{2}, t\right), \ni \alpha + \gamma = 1$, yields the result above (10f) (which shall be used throughout the present work); Notwithstanding the same physics is described when $\alpha \neq \gamma$ thusly $\hat{H}\left(i\hbar\partial_p + \alpha x, -i\hbar\partial_x + \gamma p, t\right), \ni \alpha + \gamma = 1$.

The result of course is the quantum phase space wavefunction for the quantum dynamics wave equation. Just a further comment on the $\alpha$and γ parameters in the above formulae: From the HOA ref. [1], they are otherwise arbitrary except for the condition $\alpha + \gamma = 1$. This is explained therein as a consequence of the arbitrary phase shift associated with the quantum phase space wavefunction. Further, any choice of the parameters thus constrained yields a Hamiltonian, which is dynamically equivalent (describes the same physics) as any other choice. However, it is shown therein that the Hamiltonian operator $\hat{H}\left(i\hbar\partial_p + \alpha x, -i\hbar\partial_x + \gamma p, t\right), \ni \alpha + \gamma = 1$ takes on the symmetric canonical form when $\alpha = \gamma = \frac{1}{2}$ thusly $\hat{H}\left(i\hbar\partial_p + \frac{x}{2}, -i\hbar\partial_x + \frac{p}{2}, t\right)$, $\ni \alpha + \gamma = 1$. Notwithstanding this and with an eye towards computational simplifications for particular classes of applications, it has been found that other choices than $\alpha = \gamma = \frac{1}{2}$ sometime facilitates evaluation of the integral transforms. Unless otherwise directed, the convention for α and γ shall be specified for particular cases, presently and elsewhere. For the problems herein the $\alpha = \gamma = \frac{1}{2}$ sufficeth.

Along this line, we simply note the $\alpha \neq \gamma$ version of the QPSR complex wavefunction (10g) as

$$\Psi\left(x_1,\ldots,x_n;p_1,\ldots,p_n;t\right) = \left(\begin{array}{l} \Psi_0\left(x_1,\ldots,x_n;p_1,\ldots,p_n;t=0\right) \\ \underset{\substack{\left(x_1,\ldots,x_n;\right)\\ \left(p_1,\ldots,p_n\right)}}{*}\left(\frac{1}{2\pi\hbar^2}\right)^n \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n} e^{\frac{-i}{\hbar}\sum_{j=1}^{n}\left(\begin{array}{l}\left(\gamma_j-\alpha_j\right)x_j p_j\\ +\int_0^t \hat{H}\left(\begin{array}{l}\alpha_1 x_1-\nu_{\alpha_1 x_1},\\ \ldots,\alpha_n x_n-\nu_{\alpha_n x_n};\\ \gamma_1 p_1-\nu_{\gamma_1 p_1},\\ \ldots,\gamma_n p_n-\nu_{\gamma_n p_n};u\end{array}\right)du\end{array}\right)} d\nu_{\alpha_1 x_1}\, d\nu_{\gamma_1 p_1}\cdots d\nu_{\alpha_n x_n}\, d\nu_{\gamma_n p_n} \end{array}\right) \tag{10h}$$

To recall the full details of HOA results, see the original works (refs. [2–8]). As pointed out therein,

'Notwithstanding its quantum mechanical origins, the HOA scheme takes on a life of its own and transcends the limits of quantum applications to address a wide variety of purely formal mathematical problems as well. Among other things, the result provides a formula for obtaining an exact solution to a wide variety of variable-coefficient integro-differential equations. Since the functional dependence of the Hamiltonian operator as considered is in general arbitrary upon its arguments [(i.e., independent variables, derivative operator symbols (including negative powers thereof, thus the possible integral character)], then its multivariable extension can be interpreted as the most general variable coefficient partial differential operator. Moreover, it is not confined to being a scalar or even vector operator, but may be generally construed an arbitrary rank matrix operator. In all cases of course, its rank dictates the matrix rank of the wavefunction solution.'

With these constructs, we may now grasp the nettle: the decomposition of a complex wavefunction into real and imaginary parts via the HOA (Heaviside Operational Ansatz) in the QPSR (Quantum Phase Space Representation) (viz, Eq. (2) herein); this for a CSE (Complex Schrodinger Equation) generated by a given GH (Generalised Hamiltonian) [viz, Eq. (1) herein].

3 Result

Recall that applying the scheme exhibited in Eqs. (3–10h) to the CSE of (1) yields (before incorporating the actual GH) a solution with the general form

$$\Psi\left(x_1,\dots,x_n;p_1,\dots,p_n;t\right) = \left(\begin{array}{l}\Psi_0\left(x_1,\dots,x_n;p_1,\dots,p_n;t=0\right) \\ \underset{\left(\substack{x_1,\dots,x_n;\\p_1,\dots,p_n}\right)}{\ast} \left(\frac{1}{2\pi\hbar^2}\right)^n \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n} e^{\frac{-i}{\hbar}\int_0^t \hat{H}\left(\substack{\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\dots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\\ \frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\dots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u}\right)du} d\nu_{\frac{1}{2}x_1}\, d\nu_{\frac{1}{2}p_1} \cdot\cdot d\nu_{\frac{1}{2}x_n}\, d\nu_{\frac{1}{2}p_n}\end{array}\right) \tag{11}$$

With that said, a relatively simplistic prescription results, for actually using the HOA to solve the problem:

Given the function $\hat{H}\left(\hat{x}_1,\dots,\hat{x}_n;\hat{p}_1,\dots,\hat{p}_n;t\right) = \left(\hat{H}_{re}\left(\hat{x}_1,\dots,\hat{x}_n;\hat{p}_1,\dots,\hat{p}_n;t\right)+i\hat{H}_{im}\left(\hat{x}_1,\dots,\hat{x}_n;\hat{p}_1,\dots,\hat{p}_n;t\right)\right)$ replace $\left(\hat{x}_1,\dots,\hat{x}_n;\hat{p}_1,\dots,\hat{p}_n,t\right)$ with $\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\dots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\dots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};t\right)$ as in Eq. (11). Thus the exact analytical quadrature (integral form) complex wavefunction for the QPSR version of the CSE generated by a given GH as (1).

$$\Psi\left(x_1,\dots,x_n;p_1,\dots,p_n;t\right) = \left(\begin{array}{l}\Psi_0\left(x_1,\dots,x_n;p_1,\dots,p_n;t=0\right) \\ \underset{\left(\substack{x_1,\dots,x_n;\\p_1,\dots,p_n}\right)}{\ast} \left(\frac{1}{2\pi\hbar^2}\right)^n \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n} e^{\frac{-i}{\hbar}\int_0^t \hat{H}_{re}\left(\substack{\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\\ \dots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\\ \frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\\ \dots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u}\right)+i\hat{H}_{im}\left(\substack{\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\\ \dots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\\ \frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\\ \dots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u}\right)du} d\nu_{\frac{1}{2}x_1}\, d\nu_{\frac{1}{2}p_1} \cdot\cdot d\nu_{\frac{1}{2}x_n}\, d\nu_{\frac{1}{2}p_n}\end{array}\right) \tag{12}$$

In retrospect, recall from (10) that the pre-inverted complex wavefunction $\breve{\Psi}\left(\bar{x}_1, \ldots, \bar{x}_n; \bar{p}_1, \ldots, \bar{p}_n; t\right)$ of (1) has the form

$$\begin{aligned}
&\left(\begin{array}{l}\hat{H}_{re}\left(\begin{array}{l} i\hbar\overline{p}_1+\frac{1}{2}x_1,\ldots,i\hbar\overline{p}_n+\frac{1}{2}x_n; \\ -i\hbar\overline{x}_1+\frac{1}{2}p_1,\ldots,-i\hbar\overline{x}_n+\frac{1}{2}p_n;t\end{array}\right) \\ +i\hat{H}_{im}\left(\begin{array}{l} i\hbar\overline{p}_1+\frac{1}{2}x_1,\ldots,i\hbar\overline{p}_n+\frac{1}{2}x_n; \\ -i\hbar\overline{x}_1+\frac{1}{2}p_1,\ldots,-i\hbar\overline{x}_n+\frac{1}{2}p_n;t\end{array}\right)\end{array}\right)\breve{\Psi}\left(\overline{x}_1,\ldots,\overline{x}_n;\overline{p}_1,\ldots,\overline{p}_n;t\right) \\
&= i\hbar\partial_t\breve{\Psi}\left(\overline{x}_1,\ldots,\overline{x}_n;\overline{p}_1,\ldots,\overline{p}_n;t\right) \\
&\breve{\Psi}\left(\overline{x}_1,\ldots,\overline{x}_n;\overline{p}_1,\ldots,\overline{p}_n;t\right)= \\
&\breve{\Psi}_0\left(\overline{x}_1,\ldots,\overline{x}_n;\overline{p}_1,\ldots,\overline{p}_n;t=0\right)e^{\frac{-i}{\hbar}\int_0^t\left(\hat{H}_{re}\left(\begin{array}{l} i\hbar\overline{p}_1+\frac{1}{2}x_1,\ldots,i\hbar\overline{p}_n+\frac{1}{2}x_n; \\ -i\hbar\overline{x}_1+\frac{1}{2}p_1,\ldots,-i\hbar\overline{x}_n+\frac{1}{2}p_n;u\end{array}\right)+i\hat{H}_{im}\left(\begin{array}{l} i\hbar\overline{p}_1+\frac{1}{2}x_1,\ldots,i\hbar\overline{p}_n+\frac{1}{2}x_n; \\ -i\hbar\overline{x}_1+\frac{1}{2}p_1,\ldots,-i\hbar\overline{x}_n+\frac{1}{2}p_n;u\end{array}\right)\right)du}
\end{aligned} \tag{13}$$

Equation (13) will be considered again later, 'in situ' the RSE real wavefunction auxiliary conditions.

Let us now turn to consider the RSE system (2): proceeding formally(since the GH therein is for the moment and notational simplicity, being expressed without the particular operator identifications of QPSR via (3) being written out explicitly), a separate new RSE for the real part of the complex wavefunction obtains

$$\begin{aligned}
&\left(\begin{array}{l}-\hat{H}_{im}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)\left(\frac{-\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)\Psi_{re}+\hbar\partial_t\Psi_{re}}{\hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)}\right) \\ +\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)\Psi_{re}\end{array}\right) \\
&\quad = -\hbar\partial_t\left(\frac{-\hat{H}_{im}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)\Psi_{re}+\hbar\partial_t\Psi_{re}}{\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)}\right) \\
&\quad \Psi_{im} = \frac{-\hat{H}_{im}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)\Psi_{re}+\hbar\partial_t\Psi_{re}}{\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)}
\end{aligned} \tag{14re}$$

and likewise a new separate RSE for the imaginary part of the complex wavefunction

$$\begin{aligned}
&\left(\begin{array}{l}\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)\Psi_{im} \\ +\hat{H}_{im}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)\left(\frac{\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)\Psi_{im}-\hbar\partial_t\Psi_{im}}{\hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)}\right)\end{array}\right) \\
&\quad = \hbar\partial_t\left(\frac{\hat{H}_{im}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)\Psi_{im}-\hbar\partial_t\Psi_{im}}{\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)}\right)
\end{aligned}$$

$$\Psi_{re} = \frac{\hat{H}_{im}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)\Psi_{im}-\hbar\partial_t\Psi_{im}}{\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t\right)} \tag{14im}$$

Re-arranging (14re)

$$\partial_t^2\Psi_{re} - \left(2\frac{\hat{H}_{im}\left(\hat{x}_1,\ldots,\hat{x}_n;\,\hat{p}_1,\ldots,\hat{p}_n;\,t\right)}{\hbar} + \frac{\partial_t\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\,\hat{p}_1,\ldots,\hat{p}_n;\,t\right)}{\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\,\hat{p}_1,\ldots,\hat{p}_n;\,t\right)}\right)\partial_t\Psi_{re} + \left(\begin{array}{l}\frac{\left(\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)\right)^2}{\hbar^2} + \frac{\left(\hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)\right)^2}{\hbar^2}\\ -\frac{\partial_t\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)}{\hbar}\\ +\frac{\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)\partial_t\hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)}{\hbar\hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)}\end{array}\right)\Psi_{re} = 0 \tag{15re}$$

Applying the ideas from Eqs. (3–13) to (15re) there obtains a particular QPSR operator solution for (15re) as

$$\Psi_{re_{P1}} = \cos\left(\int_0^t \frac{\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\,\hat{p}_1,\ldots,\hat{p}_n;\,u\right)}{\hbar}du\right)e^{\int_0^t \frac{\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;u)}{\hbar}du} \tag{16re}$$

Similiarly re-arranging (14im)

$$\partial_t^2\Psi_{im} - \left(2\frac{\hat{H}_{im}\left(\hat{x}_1,\ldots,\hat{x}_n;\,\hat{p}_1,\ldots,\hat{p}_n;\,t\right)}{\hbar} + \frac{\partial_t\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\,\hat{p}_1,\ldots,\hat{p}_n;\,t\right)}{\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\,\hat{p}_1,\ldots,\hat{p}_n;\,t\right)}\right)\partial_t\Psi_{im} + \left(\begin{array}{l}\frac{\left(\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)\right)^2}{\hbar^2} + \frac{\left(\hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)\right)^2}{\hbar^2}\\ -\frac{\partial_t\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)}{\hbar}\\ +\frac{\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)\partial_t\hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)}{\hbar\hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;t)}\end{array}\right)\Psi_{im} = 0 \tag{15im}$$

Applying the ideas from Eqs. (3–13) to (15im) there obtains a particular QPSR operator solution for (15im) as

$$\Psi_{im_{P1}} = \sin\left(\int_0^t \frac{\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\,\hat{p}_1,\ldots,\hat{p}_n;\,u\right)}{\hbar}du\right)e^{\int_0^t \frac{\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;u)}{\hbar}du} \tag{16im}$$

Since (15re) and (15im) are second-order linear differential operators in the time variable t, it follows that there is an additional linearly-independent solution for each (15re) and (15im). By Abel's well-known result, the second linearly-independent solution for (15re) obtains

$$\Psi_{re_{P1}} = \cos\left(\int_0^t \frac{\hat{H}_{re}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; u\right)}{\hbar} du\right) e^{\int_0^t \frac{\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;u)}{\hbar} du}$$

$$\Psi_{re_{P2}} = \Psi_{re_{P1}} \int_0^t \left(\frac{e^{\int_0^u \left(2\frac{\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;w)}{\hbar} + \frac{\partial_w \hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;w)}{\hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;w)}\right) dw}}{\left(\Psi_{re_{P1}}\right)^2} \right) du$$

$$\Psi_{re_{P2}} = \hbar \sin\left(\int_0^t \frac{\hat{H}_{re}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; u\right)}{\hbar} du\right) e^{\int_0^t \frac{\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;u)}{\hbar} du} \tag{17re}$$

Likewise for (15im)

$$\Psi_{im_{P1}} = \sin\left(\int_0^t \frac{\hat{H}_{re}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; u\right)}{\hbar} du\right) e^{\int_0^t \frac{\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;u)}{\hbar} du}$$

$$\Psi_{im_{P2}} = \Psi_{im_{P1}} \int_0^t \left(\frac{e^{\int_0^u \left(2\frac{\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;w)}{\hbar} + \frac{\partial_w \hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;w)}{\hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;w)}\right) dw}}{\left(\Psi_{im_{P1}}\right)^2} \right) du$$

$$\Psi_{im_{P2}} = \hbar \cos\left(\int_0^t \frac{\hat{H}_{re}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; u\right)}{\hbar} du\right) e^{\int_0^t \frac{\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;u)}{\hbar} du} \tag{17im}$$

Thence, by way of the constructs, particularly via Eqs. (7,8), (9) and (10) herein the Recap, (17re) [respectively (17im)] transitions to its explicit variable QPSR form as expressed in terms of its two linearly-independent solutions with their arbitrary coefficients as

$$\Psi_{re}\left(x_1,\ldots,x_n;p_1,\ldots,p_n,t\right)=$$
$$\left(\frac{1}{2\pi\hbar^2}\right)^n\times$$
$$\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n}\left(\begin{array}{l} C_{re_1}\left(\tfrac{1}{2}x_1,\ldots,\tfrac{1}{2}x_n;\tfrac{1}{2}p_1,\ldots,\tfrac{1}{2}p_n\right)\underset{\substack{x_1,\ldots,x_n;\\ p_1,\ldots,p_n}}{*} \\ \left(\begin{array}{l}\cos\left(\int_0^t\frac{\hat{H}_{re}\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\ldots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\ldots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u\right)}{\hbar}du\right)\\ \times e^{\int_0^t\frac{\hat{H}_{im}\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\ldots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\ldots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u\right)}{\hbar}du}\end{array}\right) \\ +\,C_{re_2}\left(\tfrac{1}{2}x_1,\ldots,\tfrac{1}{2}x_n;\tfrac{1}{2}p_1,\ldots,\tfrac{1}{2}p_n\right)\underset{\substack{x_1,\ldots,x_n;\\ p_1,\ldots,p_n}}{*} \\ \left(\begin{array}{l}\hbar\sin\left(\int_0^t\frac{\hat{H}_{re}\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\ldots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\ldots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u\right)}{\hbar}du\right)\\ \times e^{\int_0^t\frac{\hat{H}_{im}\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\ldots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\ldots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u\right)}{\hbar}du}\end{array}\right)\end{array}\right)$$
$$\times\ d\nu_{\frac{1}{2}x_1}d\nu_{\frac{1}{2}p_1}\cdots d\nu_{\frac{1}{2}x_n}d\nu_{\frac{1}{2}p_n} \tag{18re}$$

Respectively (17im) transitions to

$$\Psi_{im}\left(x_1,\ldots,x_n;p_1,\ldots,p_n,t\right)=$$
$$\left(\frac{1}{2\pi\hbar^2}\right)^n\times$$
$$\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n}\left(\begin{array}{l} C_{im_1}\left(\tfrac{1}{2}x_1,\ldots,\tfrac{1}{2}x_n;\tfrac{1}{2}p_1,\ldots,\tfrac{1}{2}p_n\right)\underset{\substack{x_1,\ldots,x_n;\\ p_1,\ldots,p_n}}{*} \\ \left(\begin{array}{l}\sin\left(\int_0^t\frac{\hat{H}_{re}\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\ldots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\ldots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u\right)}{\hbar}du\right)\\ \times e^{\int_0^t\frac{\hat{H}_{im}\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\ldots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\ldots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u\right)}{\hbar}du}\end{array}\right) \\ +\,C_{im_2}\left(\tfrac{1}{2}x_1,\ldots,\tfrac{1}{2}x_n;\tfrac{1}{2}p_1,\ldots,\tfrac{1}{2}p_n\right)\underset{\substack{x_1,\ldots,x_n;\\ p_1,\ldots,p_n}}{*} \\ \left(\begin{array}{l}\hbar\cos\left(\int_0^t\frac{\hat{H}_{re}\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\ldots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\ldots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u\right)}{\hbar}du\right)\\ \times e^{\int_0^t\frac{\hat{H}_{im}\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\ldots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\ldots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u\right)}{\hbar}du}\end{array}\right)\end{array}\right)$$
$$\times\ d\nu_{\frac{1}{2}x_1}d\nu_{\frac{1}{2}p_1}\cdots d\nu_{\frac{1}{2}x_n}d\nu_{\frac{1}{2}p_n} \tag{18im}$$

As mentioned earlier, (13) will now be considered again 'in situ' the RSE real wavefunction auxiliary conditions. With this in mind, the pre-inverted real wavefunction (18re) is expressed

$$\breve{\Psi}_{re}(\bar{x}_1,....,\bar{x}_n;\bar{p}_1,....,\bar{p}_n;t)=\left(\begin{array}{l}\breve{C}_{re_1}(\bar{x}_1,....,\bar{x}_n;\bar{p}_1,....,\bar{p}_n)\times\\ \left(\cos\left(\int_0^t\frac{\hat{H}_{re}\begin{pmatrix}i\hbar\bar{p}_1+\frac{1}{2}x_1,....,i\hbar\bar{p}_n+\frac{1}{2}x_n;\\ -i\hbar\bar{x}_1+\frac{1}{2}p_1,....,-i\hbar\bar{x}_n+\frac{1}{2}p_n;u\end{pmatrix}}{\hbar}du\right)\frac{e^{\int_0^t\frac{\hat{H}_{im}\begin{pmatrix}i\hbar\bar{p}_1+\frac{1}{2}x_1,....,i\hbar\bar{p}_n+\frac{1}{2}x_n;\\ -i\hbar\bar{x}_1+\frac{1}{2}p_1,....,-i\hbar\bar{x}_n+\frac{1}{2}p_n;u\end{pmatrix}}{\hbar}du}}{(2\pi\hbar^2)^n}\right)\\ +\breve{C}_{re_2}(\bar{x}_1,....,\bar{x}_n;\bar{p}_1,....,\bar{p}_n)\times\\ \left(\hbar\sin\left(\int_0^t\frac{\hat{H}_{re}\begin{pmatrix}i\hbar\bar{p}_1+\frac{1}{2}x_1,....,i\hbar\bar{p}_n+\frac{1}{2}x_n;\\ -i\hbar\bar{x}_1+\frac{1}{2}p_1,....,-i\hbar\bar{x}_n+\frac{1}{2}p_n;u\end{pmatrix}}{\hbar}du\right)\frac{e^{\int_0^t\frac{\hat{H}_{im}\begin{pmatrix}i\hbar\bar{p}_1+\frac{1}{2}x_1,....,i\hbar\bar{p}_n+\frac{1}{2}x_n;\\ -i\hbar\bar{x}_1+\frac{1}{2}p_1,....,-i\hbar\bar{x}_n+\frac{1}{2}p_n;u\end{pmatrix}}{\hbar}du}}{(2\pi\hbar^2)^n}\right)\end{array}\right) \tag{19re}$$

Likewise (18im) becomes

$$\breve{\Psi}_{im}(\bar{x}_1,....,\bar{x}_n;\bar{p}_1,....,\bar{p}_n;t)=\left(\begin{array}{l}\breve{C}_{im_1}(\bar{x}_1,....,\bar{x}_n;\bar{p}_1,....,\bar{p}_n)\times\\ \left(\sin\left(\int_0^t\frac{\hat{H}_{re}\begin{pmatrix}i\hbar\bar{p}_1+\frac{1}{2}x_1,....,i\hbar\bar{p}_n+\frac{1}{2}x_n;\\ -i\hbar\bar{x}_1+\frac{1}{2}p_1,....,-i\hbar\bar{x}_n+\frac{1}{2}p_n;u\end{pmatrix}}{\hbar}du\right)\frac{e^{\int_0^t\frac{\hat{H}_{im}\begin{pmatrix}i\hbar\bar{p}_1+\frac{1}{2}x_1,....,i\hbar\bar{p}_n+\frac{1}{2}x_n;\\ -i\hbar\bar{x}_1+\frac{1}{2}p_1,....,-i\hbar\bar{x}_n+\frac{1}{2}p_n;u\end{pmatrix}}{\hbar}du}}{(2\pi\hbar^2)^n}\right)\\ +\breve{C}_{im_2}(\bar{x}_1,....,\bar{x}_n;\bar{p}_1,....,\bar{p}_n)\times\\ \left(\hbar\cos\left(\int_0^t\frac{\hat{H}_{re}\begin{pmatrix}i\hbar\bar{p}_1+\frac{1}{2}x_1,....,i\hbar\bar{p}_n+\frac{1}{2}x_n;\\ -i\hbar\bar{x}_1+\frac{1}{2}p_1,....,-i\hbar\bar{x}_n+\frac{1}{2}p_n;u\end{pmatrix}}{\hbar}du\right)\frac{e^{\int_0^t\frac{\hat{H}_{im}\begin{pmatrix}i\hbar\bar{p}_1+\frac{1}{2}x_1,....,i\hbar\bar{p}_n+\frac{1}{2}x_n;\\ -i\hbar\bar{x}_1+\frac{1}{2}p_1,....,-i\hbar\bar{x}_n+\frac{1}{2}p_n;u\end{pmatrix}}{\hbar}du}}{(2\pi\hbar^2)^n}\right)\end{array}\right) \tag{19im}$$

So (19re) [respectively (19im)] written in abbreviated form becomes

$$\breve{\Psi}_{re}=\breve{C}_{re_1}\breve{\Psi}_{re_1}+\breve{C}_{re_2}\breve{\Psi}_{re_2} \tag{20re}$$

$$\breve{\Psi}_{im}=\breve{C}_{im_1}\breve{\Psi}_{im_1}+\breve{C}_{im_2}\breve{\Psi}_{im_2} \tag{20im}$$

Now considering boundary-value problems in the time variable t in (20re) and (20im), is expedited for the determination of the coefficients required for the arbitrary specified conditions on the real wavefunctions and their first-order time derivatives. Indeed, the appropriate coefficients may be determined via either of (20re) or (20im), and then inverse transformed through the scheme used in (10f) stated earlier herein. These inverse transformed coefficients are then substituted back into the appropriate (18re) or (18im), thus yielding exact analytical quadrature(integral form) real wavefuctions for the arbitrarily specified time auxiliary condition quantum dynamics in the QPSR.

To briefly sketch an example of this procedure, consider (20re) for initial/final value auxiliary conditions on the real part wavefunction (18re)

$$
\begin{aligned}
&\breve{\Psi}_{re}\left(t=t_{initial}\right)=\breve{C}_{re_1}\breve{\Psi}_{re_1}\left(t=t_{initial}\right)+\breve{C}_{re_2}\breve{\Psi}_{re_2}\left(t=t_{initial}\right)\\
&\breve{\Psi}_{re}\left(t=t_{final}\right)=\breve{C}_{re_1}\breve{\Psi}_{re_1}\left(t=t_{final}\right)+\breve{C}_{re_2}\breve{\Psi}_{re_2}\left(t=t_{final}\right)\\
&\breve{C}_{re_1}=\frac{\breve{\Psi}_{re}\left(t=t_{final}\right)\breve{\Psi}_{re_2}\left(t=t_{initial}\right)-\breve{\Psi}_{re}\left(t=t_{initial}\right)\breve{\Psi}_{re_2}\left(t=t_{final}\right)}{\breve{\Psi}_{re_1}\left(t=t_{final}\right)\breve{\Psi}_{re_2}\left(t=t_{initial}\right)-\breve{\Psi}_{re_1}\left(t=t_{initial}\right)\breve{\Psi}_{re_2}\left(t=t_{final}\right)}\\
&\breve{C}_{re_2}=\frac{\breve{\Psi}_{re}\left(t=t_{final}\right)\breve{\Psi}_{re_1}\left(t=t_{initial}\right)-\breve{\Psi}_{re}\left(t=t_{initial}\right)\breve{\Psi}_{re_1}\left(t=t_{final}\right)}{\breve{\Psi}_{re_1}\left(t=t_{initial}\right)\breve{\Psi}_{re_2}\left(t=t_{final}\right)-\breve{\Psi}_{re_1}\left(t=t_{final}\right)\breve{\Psi}_{re_2}\left(t=t_{initial}\right)}
\end{aligned}
\tag{21}
$$

So with the coefficients now explicitly determined in the pre-inverted form, they are then inverted to yield (again in abbreviated notation not explicitly writing out the arguments of the terms)

$$
\begin{aligned}
&C_{re_1}\left(\tfrac{1}{2}x_1,\ldots,\tfrac{1}{2}x_n;\tfrac{1}{2}p_1,\ldots,\tfrac{1}{2}p_n\right)\\
&=L^{-1}_{\begin{pmatrix}(\bar{x}_1,\ldots,\bar{x}_n)\\ \to(x_1,\ldots,x_n);\\ (\bar{p}_1,\ldots,\bar{p}_n)\\ \to(p_1,\ldots,p_n)\end{pmatrix}}\left(\frac{\breve{\Psi}_{re}\left(t=t_{final}\right)\breve{\Psi}_{re_2}\left(t=t_{initial}\right)-\breve{\Psi}_{re}\left(t=t_{initial}\right)\breve{\Psi}_{re_2}\left(t=t_{final}\right)}{\breve{\Psi}_{re_1}\left(t=t_{final}\right)\breve{\Psi}_{re_2}\left(t=t_{initial}\right)-\breve{\Psi}_{re_1}\left(t=t_{initial}\right)\breve{\Psi}_{re_2}\left(t=t_{final}\right)}\right)\\
&C_{re_2}\left(\tfrac{1}{2}x_1,\ldots,\tfrac{1}{2}x_n;\tfrac{1}{2}p_1,\ldots,\tfrac{1}{2}p_n\right)\\
&=L^{-1}_{\begin{pmatrix}(\bar{x}_1,\ldots,\bar{x}_n)\\ \to(x_1,\ldots,x_n);\\ (\bar{p}_1,\ldots,\bar{p}_n)\\ \to(p_1,\ldots,p_n)\end{pmatrix}}\left(\frac{\breve{\Psi}_{re}\left(t=t_{final}\right)\breve{\Psi}_{re_1}\left(t=t_{initial}\right)-\breve{\Psi}_{re}\left(t=t_{initial}\right)\breve{\Psi}_{re_1}\left(t=t_{final}\right)}{\breve{\Psi}_{re_1}\left(t=t_{initial}\right)\breve{\Psi}_{re_2}\left(t=t_{final}\right)-\breve{\Psi}_{re_1}\left(t=t_{final}\right)\breve{\Psi}_{re_2}\left(t=t_{initial}\right)}\right)
\end{aligned}
\tag{22}
$$

Substituting (22) into (18re), the exact analytical quadrature(integral form) real part wavefunction for the arbitrary initial/final conditions thereupon, may be expressed

$$\Psi_{re}(x_1,....,x_n;p_1,....,p_n,t)=$$
$$\left(\frac{1}{2\pi\hbar^2}\right)^n \times$$
$$\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n}\left(\begin{array}{l} L^{-1}_{\begin{pmatrix}(\bar{x}_1,....,\bar{x}_n)\\ \to(x_1,....,x_n);\\ (\bar{p}_1,....,\bar{p}_n)\\ \to(p_1,....,p_n)\end{pmatrix}}\left(\left(\frac{\begin{pmatrix}\breve{\Psi}_{re}(t=t_{final})\breve{\Psi}_{re_2}(t=t_{initial})\\ -\breve{\Psi}_{re}(t=t_{initial})\breve{\Psi}_{re_2}(t=t_{final})\end{pmatrix}}{\begin{pmatrix}\breve{\Psi}_{re_1}(t=t_{final})\breve{\Psi}_{re_2}(t=t_{initial})\\ -\breve{\Psi}_{re_1}(t=t_{initial})\breve{\Psi}_{re_2}(t=t_{final})\end{pmatrix}}\right)\right)\underbrace{*}_{\begin{pmatrix}x_1,....,x_n;\\ p_1,....,p_n\end{pmatrix}} \\ \left(\begin{array}{l}\cos\left(\int_0^t\frac{\hat{H}_{re}\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},....,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},....,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u\right)}{\hbar}du\right)\\ \times e^{\int_0^t\frac{\hat{H}_{im}\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},....,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},....,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u\right)}{\hbar}du}\end{array}\right) \\ +L^{-1}_{\begin{pmatrix}(\bar{x}_1,....,\bar{x}_n)\\ \to(x_1,....,x_n);\\ (\bar{p}_1,....,\bar{p}_n)\\ \to(p_1,....,p_n)\end{pmatrix}}\left(\left(\frac{\begin{pmatrix}\breve{\Psi}_{re}(t=t_{final})\breve{\Psi}_{re_1}(t=t_{initial})\\ -\breve{\Psi}_{re}(t=t_{initial})\breve{\Psi}_{re_1}(t=t_{final})\end{pmatrix}}{\begin{pmatrix}\breve{\Psi}_{re_1}(t=t_{initial})\breve{\Psi}_{re_2}(t=t_{final})\\ -\breve{\Psi}_{re_1}(t=t_{final})\breve{\Psi}_{re_2}(t=t_{initial})\end{pmatrix}}\right)\right)\underbrace{*}_{\begin{pmatrix}x_1,....,x_n;\\ p_1,....,p_n\end{pmatrix}} \\ \left(\begin{array}{l}\hbar\sin\left(\int_0^t\frac{\hat{H}_{re}\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},....,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},....,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u\right)}{\hbar}du\right)\\ \times e^{\int_0^t\frac{\hat{H}_{im}\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},....,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},....,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};u\right)}{\hbar}du}\end{array}\right)\end{array}\right)$$
$$\times\ d\nu_{\frac{1}{2}x_1}d\nu_{\frac{1}{2}p_1}\cdot\cdot d\nu_{\frac{1}{2}x_n}d\nu_{\frac{1}{2}p_n} \tag{23}$$

It follows that the strategy used in the above example (21–23), may be naturally extrapolated to solve other auxiliary condition problems for (18re) [respectively (18im)] as well; particularly as applied to numerous and sundry quantum control problems (e.g., chemical reactions). Consequent the findings in the present note, all of the earlier work containing the HOA complex wavefunctions in QPSR (e.g., refs. [2–8]) and the findings of others (e.g., refs. [9–11]) may now be studied via their alternative real-valued wavefunctions. MT21:42

4 Addendum: Essentials of multi-time extension

This Addendum will only state the multi-time RSE and the real part of the multi-time CSE wavefunction; these are warranted. The imaginary part analysis follows similiarly but is not included here.

$$\left(\hat{H}_{re}\left(\hat{x}_1,\ldots\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;\mathbf{t}\right)+i\hat{H}_{im}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;\mathbf{t}\right)\right)$$

$$
\begin{aligned}
&(\Psi_{re} + i\Psi_{im}) = i\hbar\partial_{\mathbf{t}} (\Psi_{re} + i\Psi_{im}) \\
&\mathbf{t} = (t_1, \ldots, t_{n-time}) \text{ multi-time} \\
&i\hbar\partial_{\mathbf{t}} = i\hbar \sum_{j=1}^{n-time} \partial_{t_j}
\end{aligned} \qquad \text{(A1multi-time)}
$$

$$
\begin{aligned}
&\hat{H}_{re}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; \mathbf{t}\right)\Psi_{re} - \hat{H}_{im}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; \mathbf{t}\right)\Psi_{im} \\
&\quad = -\hbar\partial_{\mathbf{t}}\Psi_{im} \\
&\hat{H}_{im}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; \mathbf{t}\right)\Psi_{re} + \hat{H}_{re}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; \mathbf{t}\right)\Psi_{im} \\
&\quad = \hbar\partial_{\mathbf{t}}\Psi_{re}
\end{aligned} \qquad \text{(A2multi-time)}
$$

$$
\begin{aligned}
&\left(\begin{array}{l} -\hat{H}_{im}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; \mathbf{t}\right)\left(\dfrac{-\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;\mathbf{t})\Psi_{re} + \hbar\left(\sum_{j=1}^{n-time}\partial_{t_j}\right)\Psi_{re}}{\hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;\mathbf{t})}\right) \\ +\hat{H}_{re}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; \mathbf{t}\right)\Psi_{re} \end{array}\right) \\
&\quad = -\hbar\left(\sum_{j=1}^{n-time}\partial_{t_j}\right)\left(\frac{-\hat{H}_{im}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; \mathbf{t}\right)\Psi_{re} + \hbar\left(\sum_{j=1}^{n-time}\partial_{t_j}\right)\Psi_{re}}{\hat{H}_{re}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; \mathbf{t}\right)}\right) \\
&\Psi_{im} = \frac{-\hat{H}_{im}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; \mathbf{t}\right)\Psi_{re} + \hbar\left(\sum_{j=1}^{n-time}\partial_{t_j}\right)\Psi_{re}}{\hat{H}_{re}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; \mathbf{t}\right)}
\end{aligned} \qquad \text{(1remulti-time)}
$$

Re-arranging (1remulti-time) yields

$$
\begin{aligned}
&\left(\sum_{j=1}^{n-time}\partial_{t_j}\right)\left(\sum_{j=1}^{n-time}\partial_{t_j}\right)\Psi_{re} \\
&-\left(2\frac{\hat{H}_{im}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; \mathbf{t}\right)}{\hbar} + \frac{\left(\sum_{j=1}^{n-time}\partial_{t_j}\right)\hat{H}_{re}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; \mathbf{t}\right)}{\hat{H}_{re}\left(\hat{x}_1, \ldots, \hat{x}_n; \hat{p}_1, \ldots, \hat{p}_n; \mathbf{t}\right)}\right) \\
&\quad\times\left(\sum_{j=1}^{n-time}\partial_{t_j}\right)\Psi_{re} \\
&\quad+\left(\begin{array}{l} \dfrac{\left(\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;\mathbf{t})\right)^2}{\hbar^2} + \dfrac{\left(\hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;\mathbf{t})\right)^2}{\hbar^2} \\ -\dfrac{\left(\sum_{j=1}^{n-time}\partial_{t_j}\right)\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;\mathbf{t})}{\hbar} \\ +\dfrac{\hat{H}_{im}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;\mathbf{t})\left(\sum_{j=1}^{n-time}\partial_{t_j}\right)\hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;\mathbf{t})}{\hbar\hat{H}_{re}(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;\mathbf{t})} \end{array}\right)\Psi_{re} = 0
\end{aligned} \qquad \text{(2remulti-time)}
$$

Applying the HOA to (2remulti-time) there obtains 2 particular QPSR operator solutions as

$$\Psi_{re_{P1}} = \cos\left(\int_0^{t_1} \frac{\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;\left(u,u-t_1+t_2,u-t_1+t_3,u-t_1+t_4,\ldots,u-t_1+t_{n-time}\right)\right)}{\hbar}du\right)$$
$$\times e^{\int_0^{t_1}\frac{\hat{H}_{im}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;\left(u,u-t_1+t_2,u-t_1+t_3,u-t_1+t_4,\ldots,u-t_1+t_{n-time}\right)\right)}{\hbar}du}$$

$$\Psi_{re_{P2}} = \hbar\sin\left(\int_0^{t_1} \frac{\hat{H}_{re}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;\left(u,u-t_1+t_2,u-t_1+t_3,u-t_1+t_4,\ldots,u-t_1+t_{n-time}\right)\right)}{\hbar}du\right)$$
$$\times e^{\int_0^{t_1}\frac{\hat{H}_{im}\left(\hat{x}_1,\ldots,\hat{x}_n;\hat{p}_1,\ldots,\hat{p}_n;\left(u,u-t_1+t_2,u-t_1+t_3,u-t_1+t_4,\ldots,u-t_1+t_{n-time}\right)\right)}{\hbar}du}$$

(3remulti-time)

Which in turn transitions to its explicit variable QPSR form as expressed in terms of its two linearly-independent solutions with their arbitrary coefficients as

$$\Psi_{re}\left(x_1,\ldots,x_n;p_1,\ldots,p_n,t_1,\ldots,t_{n-time}\right) =$$
$$\left(\frac{1}{2\pi\hbar^2}\right)^n \times$$
$$\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n}\left(\begin{array}{l} C_{re_1}\left(\tfrac{1}{2}x_1,\ldots,\tfrac{1}{2}x_n;\tfrac{1}{2}p_1,\ldots,\tfrac{1}{2}p_n\right) \underset{\binom{x_1,\ldots,x_n;}{p_1,\ldots,p_n}}{\underset{\smile}{*}} \\ \left(\begin{array}{l}\cos\left(\int_0^{t_1}\frac{\hat{H}_{re}\left(\begin{array}{l}\tfrac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\ldots,\tfrac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\tfrac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\ldots,\tfrac{1}{2}p_n-\nu_{\frac{1}{2}p_n};\\ \left(u,u-t_1+t_2,u-t_1+t_3,u-t_1+t_4,\ldots,u-t_1+t_{n-time}\right)\end{array}\right)}{\hbar}du\right) \\ \times e^{\int_0^{t_1}\frac{\hat{H}_{im}\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\ldots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\ldots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};\left(u,u-t_1+t_2,u-t_1+t_3,u-t_1+t_4,\ldots,u-t_1+t_{n-time}\right)\right)}{\hbar}du}\end{array}\right) \\ + C_{re_2}\left(\tfrac{1}{2}x_1,\ldots,\tfrac{1}{2}x_n;\tfrac{1}{2}p_1,\ldots,\tfrac{1}{2}p_n\right) \underset{\binom{x_1,\ldots,x_n;}{p_1,\ldots,p_n}}{\underset{\smile}{*}} \\ \left(\begin{array}{l}\hbar\sin\left(\int_0^{t_1}\frac{\hat{H}_{re}\left(\begin{array}{l}\tfrac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\ldots,\tfrac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\tfrac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\ldots,\tfrac{1}{2}p_n-\nu_{\frac{1}{2}p_n};\\ \left(u,u-t_1+t_2,u-t_1+t_3,u-t_1+t_4,\ldots,u-t_1+t_{n-time}\right)\end{array}\right)}{\hbar}du\right) \\ \times e^{\int_0^{t_1}\frac{\hat{H}_{im}\left(\frac{1}{2}x_1-\nu_{\frac{1}{2}x_1},\ldots,\frac{1}{2}x_n-\nu_{\frac{1}{2}x_n};\frac{1}{2}p_1-\nu_{\frac{1}{2}p_1},\ldots,\frac{1}{2}p_n-\nu_{\frac{1}{2}p_n};\left(u,u-t_1+t_2,u-t_1+t_3,u-t_1+t_4,\ldots,u-t_1+t_{n-time}\right)\right)}{\hbar}du}\end{array}\right)\end{array}\right)$$
$$\times\ d\nu_{\frac{1}{2}x_1}d\nu_{\frac{1}{2}p_1}\cdots d\nu_{\frac{1}{2}x_n}d\nu_{\frac{1}{2}p_n}$$

(4remulti-time)

It follows that the strategy used in the above, may be naturally extrapolated to solve other auxiliary condition problems for the multi-time RSE as well. MT21:42

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